Hierarchy in chaotic scattering in Hill's problem

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Hierarchic properties of chaotic scattering in a model of satellite encounters, studied first by Petit and Hénon, are examined by decomposing the dwell time function and comparing scattering trajectories. The analysis reveals an (approximate) ternary organization in the chaotic set of bounded orbits and the presence of a stable island. The results can open the way for a calculation of global quantities characterizing the scattering process by using tools of the thermodynamic formalism.

I. INTRODUCTION

The idea that chaotic scattering [1] may play an important role in various problems in celestial mechanics became widely accepted after the pioneering work of Petit and Hénon [2]. In their study of satellite encounters (also known as Hill's problem), they provided evidence for the presence of chaotic scattering by analyzing the dependence of a suitable parameter characterizing the outgoing asymptotic motion on the same parameter in the incoming asymptotics. They pointed out that the irregular behaviour of this scattering function, showing singularities on a fractal set, reflects the existence of scattering trajectories that are asymptotic to bounded orbits of the system.

The bounded orbits form a fractal set—called the chaotic set or nonattracting invariant set [3]—in phase space with stable manifolds extending smoothly into the region of asymptotic motion. Whenever the initial condition of a scattering trajectory is placed on a stable manifold, it gets trapped in the corresponding bounded orbit, so the structure of the singularities in the scattering function reflects the fractal structure of the chaotic set. By presenting trajectory plots of various types, Petit and Hénon underlined this structure but attempted no detailed characterization of it. In this paper, we provide a hierarchic analysis of chaotic scattering in Hill's problem by studying a slightly different scattering function, giving the dependence of the dwell time on the initial conditions, combined with a careful examination of scattering orbits. This method has recently been applied to a simple model of chemical reactions [4] making properties of the hierarchic organization clear in that problem.

At first, we give a brief introduction in Sec. II to Hill's problem based on Ref. [2], then we present the dwell time function and the basic ideas of its hierarchic analysis in Sec. III. In Sec. IV, a selection of scattering orbits is used to point out the presence of a ternary organization in the scattering data and, correspondingly, in the structure of the chaotic set. Finally, in Sec. V, we discuss the consequences of our findings including the possible use of

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the hierarchic information in the calculation of average quatities in Hill's problem or other models exhibiting chaotic scattering.

II. SATELLITE ENCOUNTERS

The behaviour of two small bodies (satellites) moving along coplanar circular orbits around a large central object (the planet) in the same (counterclockwise) direction can approximately be described, in each other's vicinity, by Hill's equations [2]:

$$\ddot{\xi} = 2\dot{\eta} + 3\xi - \frac{\xi}{\rho} \tag{1}$$

$$\ddot{\eta} = -2\dot{\xi} - \frac{\eta}{\rho} \tag{2}$$

with $\rho = \sqrt{\xi^2 + \eta^2}$. The coordinates ξ and η describe the positions of the satellites relative to each other, after an appropriate change of scales, in a comoving reference frame rotating around the planet. (In particular, if one of the satellites is much smaller than the other, then these coordinates simply give the position of the small satellite while the big one rests in the origin.) These equations contain no free parameter. They can be derived as the Hamiltonian equations of motion from a suitable function H, and, as a consequence, they give rise to a conserved quantity

$$\Gamma = 3\xi^2 + \frac{2}{\rho} - \dot{\xi}^2 - \dot{\eta}^2 \tag{3}$$

playing the role of energy in the problem.

Hill's equations allow general coplanar motions of the satellites, not just circular orbits. Neglecting the interaction between the satellites, the "free" orbits can be parametrized by the difference h of the two semimajor axes and a parameter k called the reduced eccentricity; for circular obits $k = 0$ and h is just the difference of the two orbit radii. Therefore, the family of circular initial orbits can be parametrized by a single parameter h (apart from a trivial constant that can be transformed out by a suitable choice of the origin of time), and the constant of motion Γ can be expressed through h: $\Gamma = \frac{3}{4}h^2$. The symmetries of the equations also allow us to assume $h > 0$. We will concentrate on this situation in the following and leave the case of more general initial conditions to the discussion.

If $h \neq 0$, the satellites slowly approach and get close to each other (i.e., they have an encounter) due to the difference in their periods. During the encounter, their interaction must be taken into account; as a result, they may perform complicated motion in the proximity of each other for a while, but then separate again and move away from each other asymptotically along generic elliptic orbits characterized by parameters h' and k'. Bearing in mind the Hamiltonian character of the problem, this process can be thought of as a scattering event from an incoming asymptotics with a given value of h and $k = 0$ into an outgoing asymptotics characterized by h' and k'. The complicated motion during the encounter, which gives the chaotic features to the scattering, is a manifestation of transient chaos [5], i.e., chaotic motion on a finite time scale. In Ref. [2], the function $h'(h)$ was used to point out the presence of chaotic scattering; in the next section, we use a nowadays standard approach based on the dwell time (or time delay) function.

III. THE DWELL TIME FUNCTION

In chaotic scattering processes, the dwell time function [6], measuring the time spent by a particle[1] in the region of strong interaction with the scatterer, shows irregular behaviour similar to that of other scattering data. We can construct it by choosing a one-parameter family of initial conditions and recording the dwell times for the trajectories. In our case of satellite encounters, the family of initial circular orbits parametrized by $\xi_0 = h$ is a natural choice. To give the initial conditions uniquely, we choose the origin of time so that at $t = 0$ the initial η coordinate is a fixed value $\eta_0 \gg 1$, while (as a consequence of $h > 0$ and $k = 0$) $\dot{\xi}_0 = 0$ and $\dot{\eta}_0 < 0$ is obtained from Γ. The incoming asymptotics then corresponds to moving downward along a vertical straight line with $\xi_i(t) = h$ and $\eta_i(t) = \eta_0 - |\dot{\eta}_0|t$. We choose a circle with radius R around the origin as the interaction zone and define the dwell time T as the time spent by the orbit inside this circle. The plot of the function $T(h)$ is shown in Fig. 1a.

The irregular structure of $T(h)$ appears as a set of singularities crowding in four narrow regions separated by smooth behaviour. These regions themselves, called the transition zones in Ref. [2], have a similar structure as shown by the blowup in Fig. 1b. Since the singularities correspond to trajectories asymptotically trapped by the confined orbits of the chaotic set, the self-similar pattern of singularities reflects the fractal structure of that set; thus, an analysis of the set of singularities will provide important information on the organization of the chaotic set [3,6,7]. The fractal pattern of singularities suggests that we consider the function $T(h)$ as built of a few basic blocks containing the singularities and separated by smooth regions (valleys). In turn, these basic blocks are considered again as containing smaller blocks separated by smaller regions of smooth behaviour. This approach yields a whole hierarchy of blocks sitting on top of the set of singularities: the basic blocks are on the first level of the hierarchy, the subblocks contained in them go to the second level, etc. The blocks on a given level of the hierarchy provide a coverage of the set of singularities: the higher we look into the hierarchy, the higher resolution we obtain in the coverage.

To carry out this hierarchic decomposition of the time delay function, we need rules telling us how to break the continuous picture based on the variable T into the obvious discreteness introduced by the hierarchy. These rules determining which of the singularities belong together in one block at a certain level of the hierarchy should reflect the intrinsic organization of the chaotic set. As was shown in Ref. [4] for a smooth potential model, this can be achieved by linking the rules of block construction to the topological complexity of scattering trajectories so that the orbits of the valleys separating the blocks at a given level have the same degree of complexity in their structures.

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[1]As we have noted, the particle can be identified, for simplicity, with the smaller satellite if the other one is much larger.

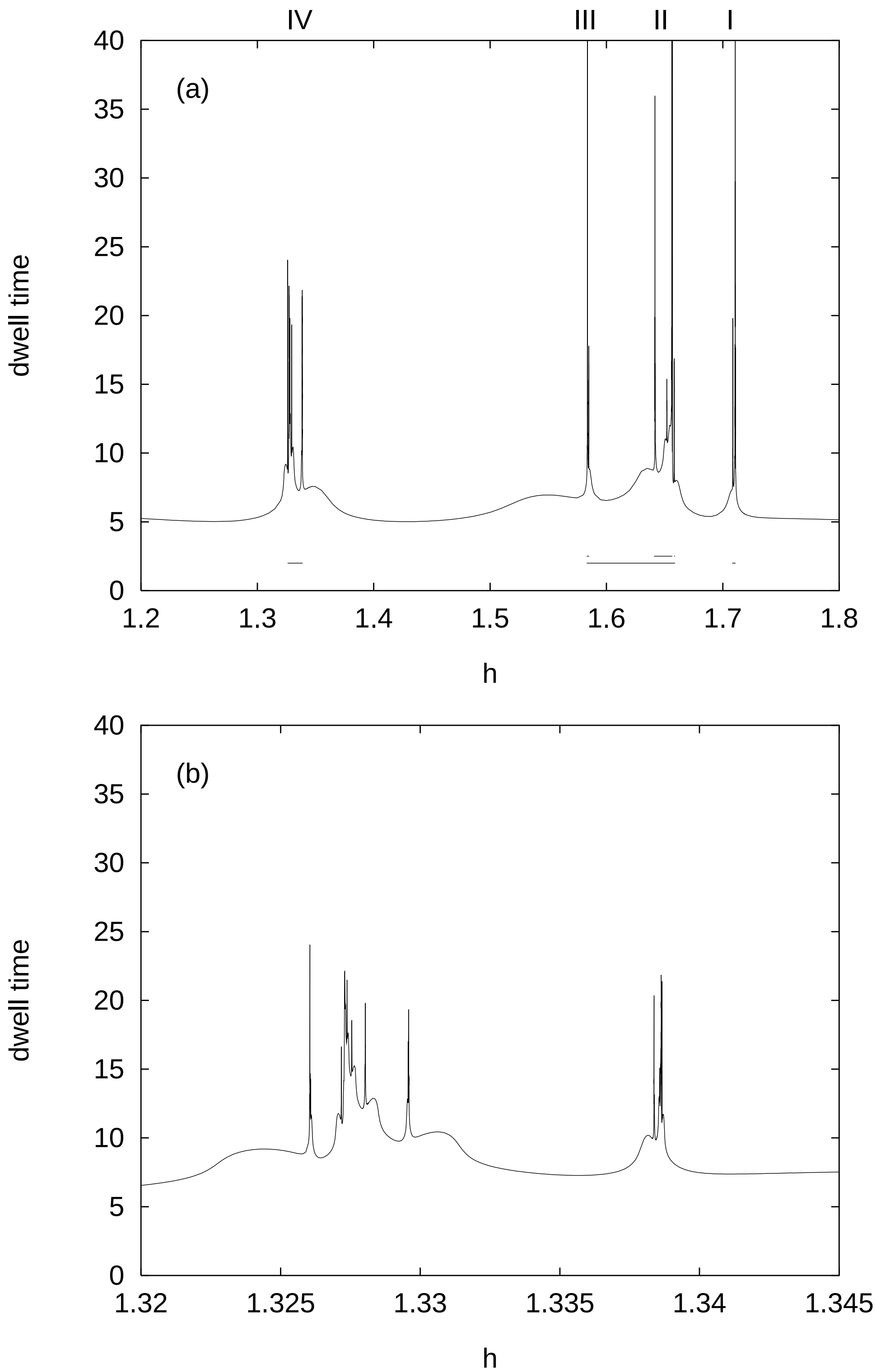

FIG. 1. (a) The dwell time function $T(h)$ for circular initial orbits with $\eta_0 = 100$ and $R = 5$ as the radius of the interaction region. The Roman numbers above the plot refer to the transition zones of Ref. [2]. The horizontal lines under the graph of $T(h)$ identify the three first-level blocks of Sec. IV and the three subblocks in one of them obtained at the second level (see text). The gap between the two subblocks on the right of the large middle block is hardly visible. (b) The blowup of zone IV.

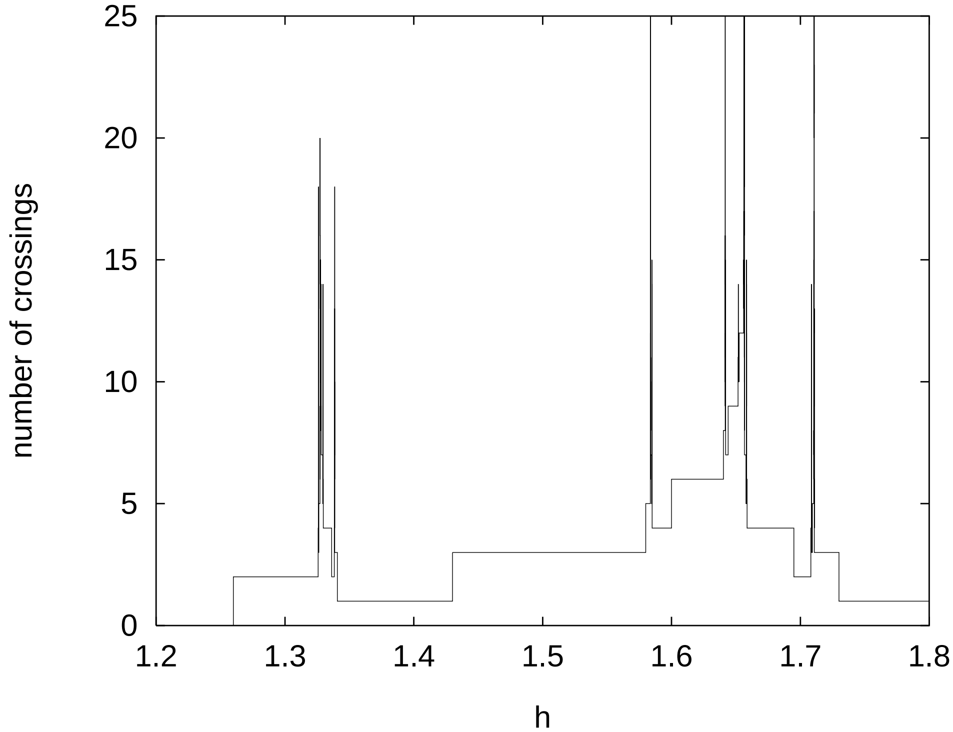

FIG. 2. The discrete dwell time function $N(h)$ for the same orbits as in Fig. 1.

One measure of the orbit complexity can be the number of crossings the orbits produce with a suitably chosen Poincaré section. The function $N(h)$ giving the number of crossing of a trajectory from our family of initial conditions with the surface $\eta = 0$ can be considered as a *discretized* dwell time function[2]: it can be shown that all the periodic orbits of the system cut this surface, so the longer a scattering trajectory follows a particular periodic orbit, the larger the number of crossings N become.

The discretized time delay function is plotted in Fig. 2; its similarity to Fig.1 is obvious. The discreteness introduced in the picture will make the decomposition process easier to implement, since the smooth changes in T are now replaced by plateaus of fixed heights that can be compared to one another. In the next section, we will obtain the rules of block construction for $N(h)$ [and thus for $T(h)$] by looking at the topological properties of scattering orbits in Hill's problem.

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[2]In principle, we should record only crossings that happen in the same direction in a Poincaré section; however, we just use the section to measure the scattering time, so this simplification will not cause any trouble.

IV. SCATTERING ORBITS AND THE TERNARY HIERARCHY

A. The topology of orbits

The scattering orbits approach, along their incoming asymptotics, the interacting region around the origin where they perform their central parts consisting of localized and usually complicated motion (for shorter or longer times, depending on the initial conditions), and finally escape downward or upward. Since it is the central part that can get close to the bounded orbits of the chaotic set, we will concentrate on the central parts of scattering orbits, ignoring differences in their escapes. In the hierarchic decomposition of $N(h)$ or any other scattering function, we would like to set apart orbits from one another according to the complexity of their central parts. For this purpose, the value N is a good indicator but we also need a visual evaluation of the graph of the orbit to make necessary distinctions between different orbit types. Our main goal is to represent the valleys of the dwell time function with scattering orbits, so that the comparison of these orbits can tell us whether two given valleys belong to the same level in the hierarchy or not (we will say that a valley is of the nth level if the shortest block containing it belongs to level $n-1$). For the comparison, we will identify some key elements in the orbit plots so that the number of these elements in the orbit will give the level number of the corresponding valley in the hierarchy.

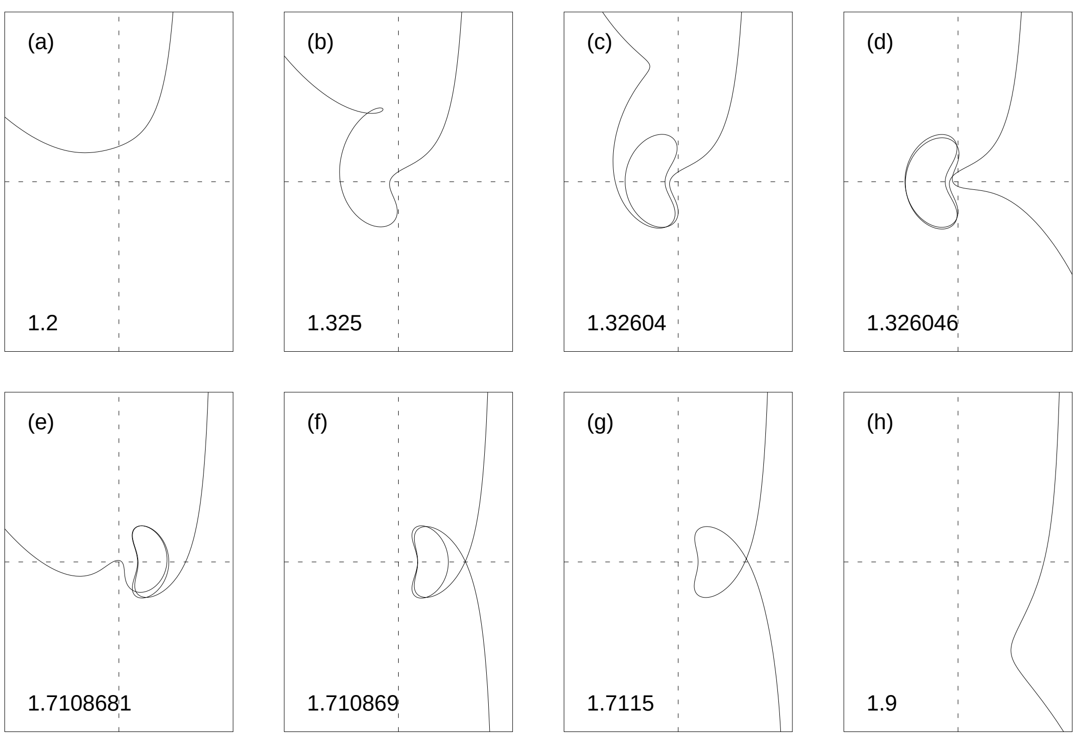

FIG. 3. The evolution of scattering orbits when approaching the outermost singularities of the dwell time function. All the windows show ξ horizontally and η vertically in the ranges $-2 < \xi < 2$ and $-4 < \eta < 4$ (with the axes as dashed lines). The numbers on the plots give the corresponding values of h in the initial conditions. The orbits in Figs. 3d and e are from inside the irregular domain.

As can be seen from Figs. 1 and 2, all the singularities of the dwell time function are contained in the domain between $h_a \approx 1.326$ and $h_b \approx 1.711$ [8]. Above h_b, all the trajectories escape downwards, the simplest ones making only one crossing with the ξ axis, on the positive ξ side (Fig. 3h). Approaching h_b from above, the orbits make more and more turns around a kidney-shaped periodic orbit (Figs. 3g and f), developed from the Lagrangean point L_2 of the system [2], before the escape. These turns appear in $N(h)$ as a sequence of shortening plateaus increasing in height in steps of two [while there is a continous divergence in $T(h)$]; the shortening is geometric governed by the stable eigenvalue of the periodic orbit. On the other side of h_b, the escape may also happen upwards (Fig. 3e). A similar phenomenon can be observed when approaching h_a from below (Figs. 3a–d): all the orbits escape upwards (the simplest ones with $N = 0$), while winding more and more around a similar periodic orbit, and for $h > h_a$ escapes can appear in both directions.

Looking at the larger valleys of $N(h)$ inside the irregular domain, we can observe that each valley is dominated by two plateaus with a difference of 2 in their N values; they also lie lowest within the valley. These plateaus correspond to the simplest possible orbits in the valley, their difference in N coming from the fact that the escaping part may cross (twice) the ξ axis. Approaching the edges of the valley, N diverges, in the same way as observed outside the irregular domain, when the trajectories get closer and closer to one of the kidney-shaped orbits but still escape in the same direction. Thus the only significant difference in the structure of two orbits from the same valley is in the number of turns around the same kidney-cycle immediately before escaping [9]. If we consider these turns as belonging to the escape process, then we can claim that central parts of all the orbits from a given valley have the same topological structure. In contrast, orbits taken from different valleys have characteristically different central parts. This property makes it possible to represent a valley with one of its scattering orbits; we will choose this from the widest plateau with a height of two above the minimum of the valley. Figure 4 shows scattering trajectories representing the largest valleys and some of the narrower ones.

B. The ternary hierarchy

Now we are in a position to start the decomposition procedure. The blocks of a given level can be obtained by removing the appropriate valleys from the blocks of the previous level. The inclusion of the final kidney-turns in the escaping part puts the border between a valley and the neighbouring block right to a singularity. To decide whether a certain smooth region between two singularities is to be deleted at a given level, we analyze the structure of its representant scattering orbit: the valleys with the simplest trajectories within the given block will be removed to yield the new subblocks. The irregular region of $N(h)$ can be separated into first-level blocks by comparing the trajectories representing the largest visible smooth regions between the transition zones of Ref. [2].

The three orbits shown in Figs. 4a, e, and b sit in the valleys separating zones IV–III–II–I, respectively. We may immediately conclude that this separation into four blocks does not satisfy the "equal complexity" requirement: the orbits of Fig. 4a and b are clearly of the same type and complexity (the central part is just one close approach of the origin), but the orbit in Fig. 4e is obviously more complicated performing two close approaches of the origin. Therefore, as a second attempt, we consider this middle valley as separating blocks

on a higher level and consequently, zones II and III as parts of one larger first-level block ranging from about 1.58 to 1.66; the regions I and IV will form the two other blocks in the first level of the hierarchy (Fig. 1a).

The fact that we obtained three blocks in the first step suggests that we should look for a ternary hierarchy dividing each block into three at the next level. Indeed, these blocks show, when magnified, an inner structure similar to the whole irregular region: there are two simplest trajectory types within each block, plotted in Fig. 4c–h. The topological similarity between the central parts of Figs. 4e and f is now obvious (two close approaches to the origin), while in the other orbits the first close approach is replaced by a prograde bend, essentially a part of the kidney-shaped orbits. Considering the close approaches and the prograde bends as basic structural elements of the central parts of trajectories, the first-level blocks contain orbits that start their central parts with one of these elements, the same element within one block. Similarly, the blocks on the second level gather together orbits starting their central parts with the same pair of basic elements, a different pair for each block. In turn, the magnification of the second-level blocks produces singularity structures resembling to the picture of Fig. 2 again, and all the 18 trajectories representing the dividing valleys can be decomposed into the three basic elements mentioned.

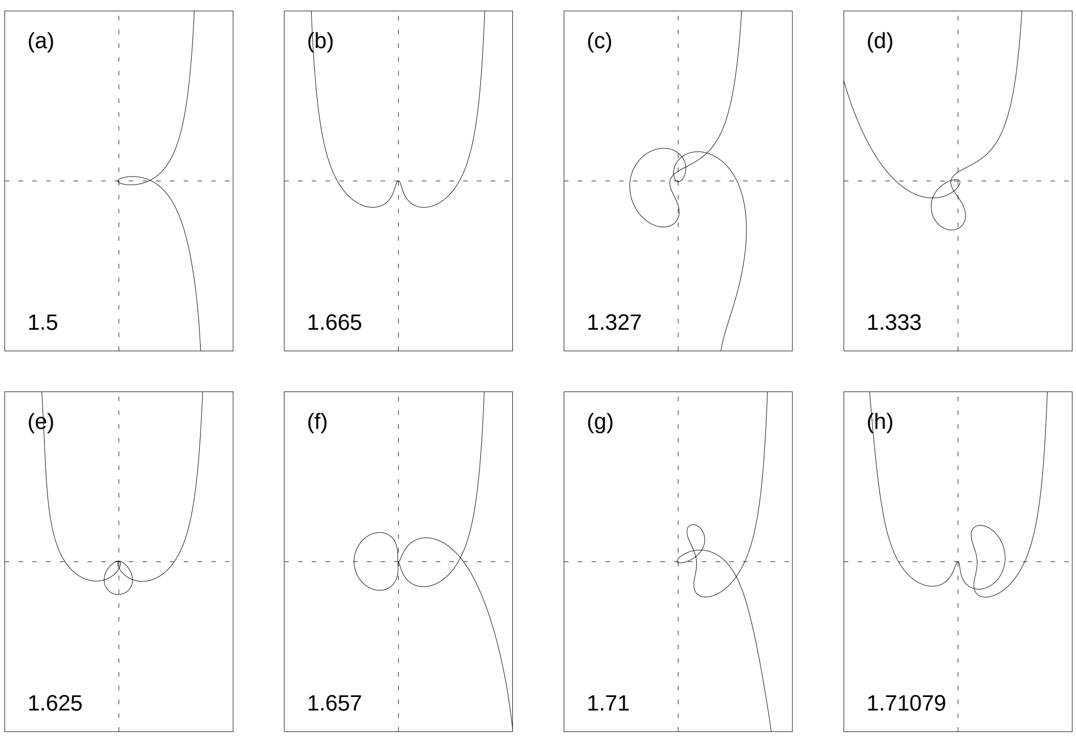

FIG. 4. The scattering orbits that represent the most prominent valleys defining the blocks on the first and second levels.

We may conclude that in general all the orbits from an nth-level block share the first n elements of their central parts, and the shortest of these orbits, with just $n + 1$ elements, sit in the valleys that will split the block into subblocks at the next level. Thus, when analyzing a scattering orbit from a given valley, the number of these elements in the central

part gives the level number of the valley and consequently the (largest) blocks it separates. The division rules can easily be formulated in terms of the discrete dwell times too: given the minimal N values for the two valleys next to a particular block, a simple rule will give the minima of the two valleys splitting that block into three at the next level. If the block intended to split sits in one side of a larger (previous-level) block next to a valley (outside the larger block) with a minimum N_0, then we look for valleys with minima $N_0 + 1$ and $N_0 + 2$ inside; however, if the block sits in the middle with neighbouring valley minima N_1 and N_2, then the new valleys are those with minima $N_1 + 3$ and $N_2 + 3$.

C. The stable island

The ternary organization established by the rules above can be followed climbing higher in the hierarchy, and we found that it is complete up to the fifth level. In constructing the sixth level of the hierarchy, the rules of the decomposition cannot be applied to split one block in the center of the level since it has an inner structure different from that of the other blocks (Fig. 5). This particular block contains trajectories that are built of only close approaches to the origin avoiding the kidney-shaped orbits, and are organized in a way which is not captured by the rules we used before.

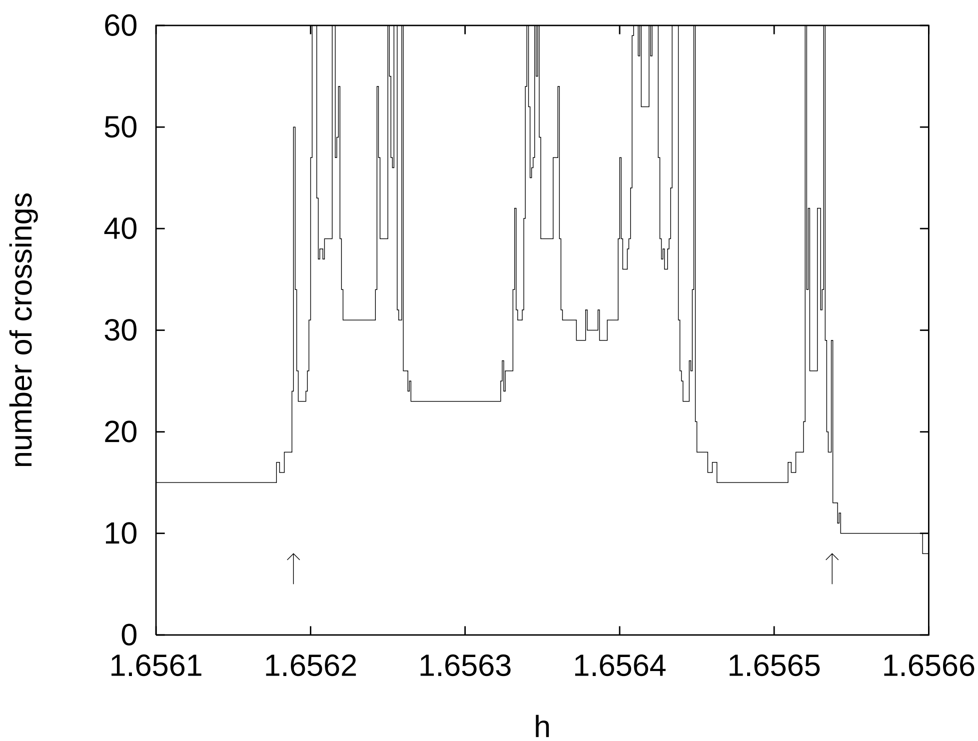

FIG. 5. The magnification of the fifth-level block with an anomalous inner structure. Notice the difference between this plot and Fig. 2. The arrows at values h_{3a} and h_{3b} mark the positions of the two scattering orbits running directly into the 3-cycle.

The reason for this anomaly is that these orbits are close to a region of *stable motion* corresponding to a bound configuration of the two satellites. In the Poincaré section, the stable motions form a KAM island with the well-known hierarchy of periodic orbits with chains of smaller islands around the edge, and the rules of organization for that hierarchy differ from those of the ternary organization. Although the scattering trajectories cannot reach the inside of the island, the complicated structure around the edge of the island results in scattering trajectories that are trapped for anomalously long times in its vicinity (Fig. 6a). The existence of these anomalous scattering orbits was noted in Ref. [2]; now we can explain it by the presence of the island.

The image of the primary "stable" block reappears in the middle of other "ordinary" blocks at higher levels of the hierarchy copied there by the dynamics. In general, keeping away from the two kidney-shaped orbits for a certain number of steps in the refinement process of any block in the hierarchy will reveal these island-approaching orbit blocks with a non-ternary inner structure. On the other hand, the subtleness of these flaws in the rules describing the hierarchy also means that the ternary organization remains a good approximation to the hierarchic properties everywhere in the chaotic set except in a narrow region around the island.

It is important to note that near the two edges of the primary stable block there are two values h_{3a} and h_{3b} of the initial condition for which the scattering trajectory runs directly into an eight-shaped hyperbolic orbit with $N = 6$ and two close approaches of the origin, avoiding the kidney-cycles (Fig. 6b). In a proper Poincaré section, where we record only crossings with e.g. $\dot{\eta} > 0$, this orbit appears as a period-3 cycle. In fact, the close approaches as basic elements of orbit structure connected to the rules of the ternary hierarchy reflect encounters with this cycle. In other words, the structure of the family of scattering orbits is organized not only by the two kidney-cycles, as claimed in Ref. [2], but also by this 3-cycle representing the island. This also means that the chaotic set itself is built on these three orbits as its pillars.

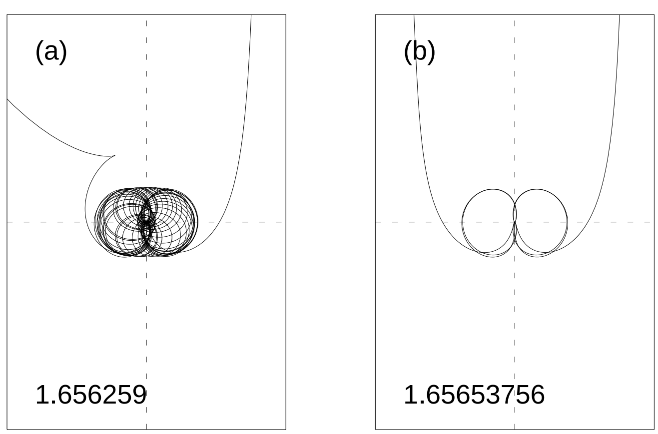

FIG. 6. (a) One of the scattering orbits from inside the stable block in Fig. 5 with anomalously large dwell times ($N = 108$ for this one). (b) A scattering trajectory with $h \approx h_{3b}$ approaching the period-3 cycle in the vicinity of the stable island.

The 3-cycle lies very close to the island, and its stable and unstable manifolds form a "cage" around it: the orbits that enter the cage form the irregular inner structure of the stable blocks. The presence of the cage also "hides" the island for scattering trajectories as long as they stay away from its immediate vicinity: the irregularities caused by the island in the hierarchy are covered by the hyperbolic behaviour of the neighbouring 3-cycle. The closeness of the period-3 orbit to the island is a consequence of the delicate interplay known as *squeezing* between a stable island around a fixed point and a nearby period-3 resonance [10] found in generic Hamiltonian systems with a quadratic nonlinearity.

V. DISCUSSION

We have shown that the singularity structure appearing in the dwell time function in Hill's problem can be decomposed into blocks ordered in a ternary hierarchy which remains a good approximation even in the presence of the distorting effects of a stable island. The main tool of the decomposition is the analysis of the topological structure of scattering trajectories: it can provide, in conjunction with the discretized dwell time function $N(h)$, the information we need on the hierarchic properties. In the simplest examples of chaotic scattering like the three-disk model, the discretized dwell time function in itself can reveal the hierarchic structure immediately [7]: cutting its graph at a height n will yield all the blocks in the nth level. However, in more complicated cases like the model discussed here, the structure of the graph is less revealing, so we need the extra information embodied in the scattering orbits. The orbit plots we used in our analysis were mostly the same as produced in Ref. [2], but we surveyed them systematically to uncover the details of the hierarchy.

This hierachic analysis is not restricted to the dwell time function: it can be carried out on any scattering function (e.g. $h'(h)$ used in Ref. [2]) that mirrors in its singularities the structure of the chaotic set. We have chosen the dwell time function $T(h)$ because of its close relationship with $N(h)$. The analysis of Hill's problem reported here also provides another example illustrating the capabilities of the hierarchic analysis in models of chaotic scattering with two degrees of freedom.

The hierarchic information can be used in calculations of various global quantities characterizing the scattering process. The main tool for such calculations is the thermodynamic formalism [11], where the lengths $l_i^{(n)}$ of blocks at level n of the hierarchy are put into the scaling sum [7]

$$\sum_i \left(l_i^{(n)}\right)^\beta \sim e^{-\beta F(\beta) n} \tag{4}$$

to produce the free energy function $F(\beta)$. Specific values of this function yield a few important scaling quantities [7]. The most notable ones are: (i) the topological entropy $K_0 = -\beta F(\beta)|_{\beta=0}$ characterizing the growth rate of the numbers of blocks from one level to the next ($K_0 = \log 3$ for a ternary hierarchy), (ii) the fractal dimension D_0 of the set of singularities [from $F(D_0) = 0$], and (iii) the escape rate $\kappa = F(1)$ describing the exponential decay of survival of scattering trajectories within the interacting region. The number κ also describes the exponential shrinking of the total length of blocks at a given level if we go higher and higher in the hierarchy. In principle, the nonhyperbolic contributions due to the island produce nonexponential scalings, leading to $D_0 = 1$ and $\kappa = 0$, but the screening

effect of the hyperbolic 3-cycle usually ensures that the scaling remains exponential for levels below a certain crossover value.

The hierarchic structure can be exploited to obtain reliable results for integrals like $I = \int_0^\infty h\Delta(h^2)\, dh$ (with $\Delta(h^2) = h'^2 - h^2$) describing the rate of energy transfer in a planetary ring system composed of many particles of equal mass [2]. Because of the irregular behaviour of the scattering function $h'(h)$ the integral cannot be determined very accurately by a usual algorithm—a fact that has already been noted in Ref. [2]. However, we can use the hierarchic information to make the calculations more consistent [4]: we can evaluate the integral within the smooth valleys separately and add up their contributions in the order they appear in the hierarchy, starting from the lowest level. After the first n levels, the regions still missing from the integral are the blocks at level $n + 1$ and higher. Since their total length decays as $\exp(-\kappa n)$, our procedure converges exponentially fast (as long as we can neglect the nonhyperbolic effects associated with the island).

Concerning the validity of our findings, the particular choice of circular initial orbits ($k = 0$), used in Ref. [2] and in our study too, is not special in the sense that a wide choice of the possible combinations of the parameters in the general form of the asymptotic motion can lead to similar results. The only requirement is that the line representing the initial conditions in phase space cut the stable manifolds of the chaotic set [7]. As we have seen, the ternary structure found in our case rests on the two kidney-shaped orbits and the period-3 cycle near the stable island of bound motion. These basic orbits can be found in Hill's problem with the same properties for a certain range of Γ values, so if our set of initial conditions remains within that range, then we find the same organization (but, of course, with different scaling ratios).

Finally, it is worth noting that the presence of the stable island remains the main difference between Hill's problem and the inclined billiard model [12], proposed as a simple alternative with similar properties, where a ball bounces on two large overlapping disks in a gravitational field. The hierarchic organization found there was simply binary built around the two fixed points of vertical bouncing (analogous to the kidney-cycles). We can make the structure of this billiard model more resembling to that of Hill's problem by smoothing out the corner where the disks meet: then there is a third fixed point which can be elliptic if the curvature of the dip is small enough so that the resulting stable island provides the third structural element in the organization of the scattering orbits.

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